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A System of Biquadratic Diophantine Equations

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Abstract—*In this paper, I solve* $x^4 + y + 1 - xyz = 0$,$x^4 + y^3 - y^2 + y - 1 + xyz = 0$*completely and study a pair of simultaneous biquadratic Diophantine equations (1)* $x/(y^4 + 1)$ *and* $y/(x^4 + 1)$, *where x and y are positive integers. The main result in this paper is that there exist an infinite number of sequences such that x and y satisfy (1) if and only if they are consecutive terms of one of these sequences.*

1. INTRODUCTION

Many interesting results have been obtained for the equation $z = \frac{g2(x,y)}{g1(x,y)}$where $g1(x, y)$ and $g2(x, y)$ are special quadratic polynomials. These are due to Barnes, Goldberg, and Mills. There may be an infinity of comparatively trivial solutions but only a finite number of possible values for z. For this z we have a quadratic equation z $g2(x, y)$= $g1(x, y)$.The values of x and y are either finite in number or, if not can be found from a Pell equation or by a recursive algorithm. The equation $x^2 + y^2 + 1 - xyz = 0$, equivalently x/y^2 +1 and y/x^2+1 has positive integral solutions (x,y) = (u_n , u_{n+1}) where u sequence is ….,13, 5, 2, 1, 1, 2, 5, 13, ………. With z= 3. This sequence consists of alternate terms of the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13,….The equation $x^2 + y^2 + x + y + 1 = xyz$, x>0 , y>0 has solution (x,y), the two consecutive terms of sequence 1, 1, 3, 15, …. Where u_n =5 u_{n-1} - u_{n-2} -1, while the equation $x^2 + y^2 - x - y + 1 = xyz$, x>0, y>0 has only solution x = y = 1.

Mordell [6] has shown that $ax^3 + by^3 + c = xyz$, where a, b,c are integers and has an infinite number of solutions with (x,y)=1. The solutions (x,y) can be given as polynomials in a,b,c. In this paper I find all integrals solutions for the equation $x^4 + y + 1 - xyz = 0$. I also prove that the equation $x^4 + y^3 - y^2 + y - 1 + xyz = 0$, has an infinite number of solutions in positive integers. Finally I have shown the existence of an infinite number of sequences such that x ,y satisfy $x/(y^4 + 1)$ and $y/(x^4 + 1)$, if and only if they are consecutive terms of one of these sequences.

DEFINITION. A sequence of positive integers { u_i} with at least three terms u_1 , u_2, u_3 ,….. in which any three consecutive terms satisfy the relation u_{n-1} . u_{n+1} = u_n ^{4}+1, is called a 1 chain. I will consider two chains { u_i}, { v_i } same iff there exists an h such that either u_n = v_{h-n} for all n.

THEOREM 1. Two positive integers x and y satisfy $x/(y^4 + 1)$ and $y/(x^4 + 1)$, iff they are consecutive terms of a 1 chain. Furthermore, any two consecutive terms of a 1 chain determine it completely.

Proof. Let x and y be two positive integers satisfying $x/(y^4 + 1)$ and $y/(x^4 + 1)$ (1)

Then there exists a unique positive integer r such that xr = $y^4 + 1$. Now xr ≡ $1\ (mod$ y) and hence $x^4 \equiv x^4(r^4 + 1) \equiv x^4 + 1 \equiv 0(\mathrm{mod}\ y)$.Furthermore,(x,y) =1 implies that $(r^4 + 1) \equiv 0(mod\ y)$. Thus y/ $r^4 + 1$ and r/ $y^4 + 1$ (2)

Continuing like this we obtain a sequence ……., x, y,r. such that any two consecutive terms u_n , u_{n+1} of this sequence satisfy u_n /u^4_{n+1} +1 and u_{n+1} /u^4_n +1 and any three consecutive term u_{n-1}, u_n , u_{n+1} satisfy

u_{n-1}. u_{n+1} = u_n ^{4} +1. Hence x and y are consecutive terms of a 1 chain. From the above discussion it is clear that any two consecutive terms of a 1 chain determine it completely. Converse is discussed on same lines.

2. ANALYSIS

THEOREM 2. The solutions of the Diophantine equation $x^4 + y + 1 - xyz = 0$in integers are given by (x,y,z) = (1,1,3) and (-1,-2,0).

Proof. Consider equation $x^4 + y + 1 - xyz = 0$, where x,y ,z are integers we have x/y+1 and y/x^4 +1 this implies xy /(x^4 +1)(y+1)= x^4 y+ x^4 +y+1 whence xy/ x^4 +y+1, hence ther exists a positive integer z such that x^4 +y+1 = xyz or x^4 +y+1 – xyz =0. For finding solution of x^4 +y+1 – xyz =0, we must solve system x/y+1and y/ x^4 +1.Suppose that x/y+1and y/ x^4 +1 then we have two positive integers r and s such that

rx= y+1 (3) and

sy = x^4 +1 (4)

From (3) and (4) it implies s.(rx-1) = x^4 +1 (5),

Write (5) as x(rs-x^3)= s+1

Put rs-x^3 =n, we get xn =s+1 then

x^3 = rs– n = r(xn-1)-n = rxn-r-n = rxn-(n+r) (6)

now (6) implies x (x^2 -rn) = -(n+r)

- ⇨ $rn > x^2$
- ⇨ $rn > x$

suppose $rn = x^2 + k$, where k is a positive integer

put $rn = x^2 + k$ in (6)

$x^3 = x (x^2 + k) - (n+r)$,

$xk = n+r$ then it follows

$(n-1)(r-1)+(x^2 -1)(k-1)= 2$

$nr-n-r+1+ x^2 k- x^2 -k+1=2$

$x^2 k+2-xk=2$

$xk(x-1) +2 = 2$, provided $xk(x-1) = 0 \Rightarrow x=0$, 1 but $k \neq 0$ because k is positive integer ,thus three possibilities are there

$(n-1) (r-1) = 0$ or $(x^2 -1) (k-1) =2$ (7)

$(n-1) (r-1) = 2$ or $(x^2 -1) (k-1) =0$ (8)

$(n-1) (r-1) = 1$ or $(x^2 -1) (k-1) =1$ (9)

From equation (7) n=1 or r=1 and $x^2 -1 =2$ or $k-1 = 1 \Rightarrow x^2 = 3$, not possible or if $x^2 -1 =1$ or $k-1 = 2 \Rightarrow x^2 = 2$, notpossible, so from equation (7) no solution exists.

From equation (8)

$(n-1) (r-1) =2$ and $(x^2 -1) (k-1) =0$

n-1=2 or r-1=1 and $x^2 -1=0$ or k-1=0

or n-1 =1, r-1=2 and $x^2 =1$ or k=1

n = 3 or r = 2 and x = ±1 or k = 1

Case 1. When x =1 and k=1 then $x^2 +k = 2$ but $x^2+k =rn \Rightarrow rn = 2 \Rightarrow r=1, n=2$ or n=1, r=2

But $y = rx-1 \Rightarrow y = 1x1-1 =0$ or $y = 2x1-1 =1$,so we get (x,y) = (1,0) ,(1,1) put (x,y) = (1, 0) in equation $x^4 + y + 1 - xyz$ =0 => 1+0+1-0 = 0 i.e. 2 = 0 incompatible situation so rejected.

Put (x,y) = (1,1) in equation $x^4 + y + 1 - xyz$ =0 then 1+1+1-z=0 => z= 3 ,so only solution is (1,1,3)

Case 2. When x = -1 then $x^2+k =2 \Rightarrow r=1, n=2$ or n=1 , r= 2

Subcase 1. When x = -1 and r = 1 then $y =rx-1 \Rightarrow y = 2$ so from equation $x^4 + y + 1 - xyz$ =0, we have 1-2+1-2z =0 => z=0 so (-1, -2,0) is another solution.

Subcase 2. When x = -1 and r = 2 then $y = rx-1 \Rightarrow y= -3$ so from equation $x^4 + y + 1 - xyz$ =0, we have 1-3+1 -3z =0 => z = -1/3 so rejected.

From equation (9)

$(n-1) (r-1) = 1$ and $(x^2 -1) (k-1) =1$

n-1 =1, r-1 = 1 and $x^2 -1 =1$, k-1=1

n=2, r =2 and $x^2 =2$, k = 2 not possible so rejected.

THEOREM 3. The Diophantine equation $x^4 + y^3 - y^2 + y - 1 + xyz = 0$; x >0, y> 0 has an infinite number of integral solutions.

Proof. Let $x = x_1$ and $y = y_1$ be a positive integral solution of given equation. Let x = 1, y =1 is one such solution. It is easy to see that (x,y) is a solution if and only if $x|y^3 - y^2 + y - 1$ and $y|x^4 - 1$.

Claim. If (x_1, y_1) is a solution then (x_2 , y_2) is another solution where $x_2 = (y^3 - y^2 + y - 1) | x_1$ and $y_2 = (x^4 - 1)| y_1$. Clearly (x_2, y_2) is different from (x_1, y_1)

Since $x_1 |y_1^3 - y_1^2 + y_1 -1$ and $y_1 |x_1^4 -1$

We have $x_2 = y_1^3 - y_1^2 + y_1 -1/ x_1$ an integer.

Further $x_2 | y_1^3 - y_1^2 + y_1 -1$ (11)

whence $(x_2, y_1) =1$

From $x_2 x_1 \equiv 1 \pmod{y_1}$

We get $x_1^4 (x_2^4 +1) \equiv (x_1 x_2)^4 +(x_1)^4 \equiv (x_1)^4 +1 \equiv 0 \pmod{y_1}$

Since $(x_2 , y_1)=1$, it follows that $y_1 |x_2^4 +1$ and hence $y_2 = x_2^4 +1/y_1$ is an integer and $y_2 | x_2^4 +1$

Again $y_1 y_2 \equiv 1 \pmod{x_2}$

- ⇨ $y_1^3 (y_2^3 - y_2^2 + y_2 -1) \equiv y_1^3 y_2^3 - y_1^3 y_2^2 + y_1^3 y_2 - y_1^3 \equiv 1- y_1 + y_1^2 - y_1^3 \equiv 0 \pmod{x_2}$, using equation (11)
- ⇨ Since $(x_2 , y_1)=1$ and $x_2 | y_2^3 - y_2^2 + y_2 -1$
- ⇨ Hence (x_2 , y_2) is a solution.

- ⇨ COROLLARY 3.1. If (x_1 , y_1) is a solution of equation $x^4 + y^3 - y^2 + y - 1 + xyz = 0$; x > 0 ,y > 0 then (x_0 , y_0) is also a solution where $x_0 = y_0^3 - y_0^2 + y_0 -1 / x_1$ and $y_0 = x_1^4 +1/ y_1$

EXAMPLE. Take (1,1) as solution of $x^4 + y^3 - y^2 + y - 1 + xyz = 0$, then using theorem 3

$x_2 = y_1^3 - y_1^2 + y_1 -1/ x_1 = 0$ and $y_2 = x_2^4 +1/ y_1 = 2$

From (1,1) we get (0,2) as solution and using corollary3.1 we get $x_0 = y_0^3 - y_0^2 + y_0 -1/ x_1$ and

$Y_0 = x_1^4 +1/ y_1 \Rightarrow y_0 = 2$ and $x_0 = 5$. Thus $(x_0 , y_0)=(5,2)$is a solution obtained from (1,1). Therefore repeated application of theorem 3 and corollary 3.1 yield <- (5,2) <- (1,1)->(0,2)->

OPEN PROBLEM. Is there exist any pair (x,y) such that $x|y^3 - y^2 + y - 1$ and $y|x^4 - 1$, which can be obtained from (1,1) by using theorem 3 and its corollary.

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